Comparing Normal Curve Values Using z-scores

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Standard "scores" (or z-scores) for normal data

Assume X = N(mean, std dev), that is to say, X data is normally distributed with the given statistics.

The Standard "score" (or z-score) is

$$Z = \frac{X - Mean}{Std\ Dev}$$

It counts "standard deviations above or below the mean".

If Bio test scores are B = N(75,5), a score of $b = 75$ results in a stardard score of $z_B = \frac{b - Mean_B}{Std\ Dev_B} = \frac{75-75}{5} = 0$. (50th percentile.)

If Chem test scores are C = N(80,4), a score of $c = 76$ results in a standard score of $z_C = \frac{c - Mean_C}{Std\ Dev_C} = \frac{76-80}{4} = -1$. (16th percentile.)

If Bio test scores are B = N(75,5) and Chem test scores are C = N(80,4), and you got 78 on Bio and 82 on Chem, which is better?

Bio: a specific score of $b = 78$ results in a stardard (or z-score) for Bio of $z_B = \frac{78-75}{5} = 0.6$. (No clue what percentile!)

Chem: a specific score of $c = 82$ results in a z-score for Chem of $z_C = \frac{82-80}{4} = 0.5$. (No clue what percentile!)

Which is better and why? The key is to compare the z-scores.

Your Bio test result (though numerically lower than your Chem test result) is more impressive because it has a higher z-score!

The Bio result corresponds to a higher percentile than the Chem one, even though we don't know either of these values.

KEY POINT: *z-scores allow us to "compare apples and oranges", by "putting on a level playing field" numbers from that arise in different contexts.* We just compare the appropriate z-scores.

Again assume Bio test scores are B = N(75,5) and Chem test scores are C = N(80,4). Your friend Danielle got 74 on Bio and 81 on Chem, which test did she do better on?

A Bio score of $x = 74$ results in $z_B = \frac{74-75}{5} = -0.2$.

A Chem score of $y = 81$ results in $z_C = \frac{81-80}{4} = 0.25$.

Comparing z-scores, we see that Danielle did better on Chem: she did above average on that subject, but below average on Bio!

Now consider your cousin Ed: he got 72 on Bio and 78 on Chem, which test did he do better on?

A Bio score of $b = 72$ results in $z_B = \frac{72-75}{5} = -0.6$.

A Chem score of $c = 78$ results in $z_C = \frac{78-80}{4} = -0.5$.

Comparing z-scores, we see that Ed also did better on Chem: -0.5 is a bigger z-score than -0.6. (Chem score is less bad than the Bio)

Car Price Example–and a New Question Type

Assume the price of a certain type of Mercedes is M = N(35,3) and the price of a certain type of Audi is A = N(26,4) (both in thousands of dollars).

1. Which is a better deal, paying $36K for a Merc or paying $28K for an Audi?

Again, we compare z-scores, but beware: "better deal" here doesn't mean higher value, it means lower value (less money to pay).

Note that $m = 36$ results in $z_M = \frac{36-35}{3} = 0.3333$. Also, $a = 28$ results in $z_A = \frac{28-26}{4} = 0.5$.

The Merc is a better deal, because it has a lower z-score. You're overpaying (compared to the average) in both cases, but you're overpaying by more for the Audi, so that's a worse deal!

Car Price Example–and a New Question Type

2. Which is a better deal, paying \$33K for a Merc or paying \$23K for an Audi?

This time $m = 33$ results in $z_M = \frac{33-35}{3} = -0.6667$. Also, $a = 23$ results in $z_A = \frac{23-26}{4} = -0.75$.

The Audi is a better deal, because it has a lower z-score. You're paying under average price in both cases, but you're underpaying by more for the Audi, so that's a better deal!

3. What Merc price corresponds to an Audi price of \$30K?

If you pay \$30K for the Audi, then $z_A = \frac{30-26}{4} = 1$. That's exactly 1 std dev above the mean price (84th percentile). So the answer is 1 std dev above the mean price for the Merc, namely \$35K + \$3K = \$38K. Note that $m = 38$ yields $z_M = \frac{38-35}{3} = 1$ too.

Paying at the same level means having the same z-scores.

Car Price Example–and a New Question Type

4. What Audi price corresponds to a Merc price of \$33K?

If you pay \$33K for the Merc, then $z_M = \frac{33-35}{3} = -0.6667$. That's exactly 2/3 of a std dev below mean price (unknown percentile). So the answer is 2/3 of a std dev below the mean price for the Audi, namely $\$26K - (2/3)\$4K = \$23.3333K$. Note that indeed $a = 23.3333$ does yield $z_A = \frac{23.3333-26}{4} = -0.6666$.

Paying at the same level means having equal z-scores, suggesting another way to do this problem: set the two z-scores equal (using A for the to-be-found Audi price) and solve for A using algebra.

Write

$$\frac{A-26}{4} = \frac{33-35}{3} = -0.6667.$$

So, $A - 26 = -(0.6667)(4)$ and $A = 26 - (0.6667)(4) = 23.3333$.

Comparing Heights–and saying goodbye to easy numbers

Assume that in Georgia adult men's heights (in inches) are N(69.2,3.1) and that adult women's heights are N(64.5,2.6).

1. Who's taller, relative to their gender group, a 66 inch tall woman or a 71.5 inch tall man?

The woman's z-score is $z_W = \frac{66-64.5}{2.6} = 0.5769$. The man's z-score is $z_M = \frac{71.5-69.2}{3.1} = 0.7419$. The man is taller, relatively, his height corresponding to a higher (but unknown) percentile.

2. Repeat, for a 63.5 inch tall woman and a 67 inch tall man.

The woman's z-score is $z_W = \frac{63.5-64.5}{2.6} = -0.3846$. The man's z-score is $z_M = \frac{67-69.2}{3.1} = -0.7097$. The woman is taller, relatively, her height corresponding to a higher (but unknown) percentile.

Comparing Heights–and saying goodbye to easy numbers

3. How tall would a woman need to be, to be at the same height level for women as a 6 foot tall man is for men? 6 ft = 72 in.

We set the z-scores equal (using W for the to-be-found Woman height) and solve for W using algebra.

We write

$$\frac{W - 64.5}{2.6} = \frac{72 - 69.2}{3.1} = 0.9032.$$

So, $W - 64.5 = (0.9032)(2.6)$ and $W = 64.5 + (0.9032)(2.6)$ $= 66.8484$ inches.

Comparing Heights–and saying goodbye to easy numbers

4. How tall would a man need to be, to be at the same height level for men as a 6 foot tall woman is for women?

This time we write

$$\frac{M - 69.2}{3.1} = \frac{72 - 64.5}{2.6} = 2.8846.$$

Now we solve for M. We find that $M - 69.2 = (2.8846)(3.1)$ and $M = 69.2 + (2.8846)(3.1) = 78.1423$ inches.