Introduction to Probability

Colm Mulcahy

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Probability as Likelihood as “longterm frequency”

When we say that 16% of babies arrive at least 2 weeks early, we mean that if we looked at arrival data for thousands of randomly selected babies, we’d expect about 16% of them to have arrived at least 2 weeks early. We might say that “the likelihood” of a baby arriving that early is 16%. Of course, we don’t expect that out of every 100 babies exactly 16% of them arrived that early!

Similarly, if we toss a fair coin, we expect it to come up Heads about 50% of the time, on average. We don’t expect that out of every 100 tosses of a fair coin exactly 50 come up heads. But while 4 heads out of 10 tosses is not unreasonable, 4 million heads out of 10 million tosses seems unlikely: perhaps the coin is not fair.

For a fair coin we write $P(H) = 0.5 = \frac{1}{2}$ and say “the probability of Heads is 0.5”. Similarly coin we write $P(T) = 0.5 = \frac{1}{2}$ and say “the probability of Tails is 0.5”. Note that $P(H) + P(T) = 1$.

Fair Coins and Dice

Tossing a fair coin leads to 2 **Equally Likely Outcomes** (ELOs), namely, $\{H, T\}$, each of which has probability $0.5 = \frac{1}{2}$. If our focus is on counting Heads instead, we might list those possible outcomes as $\{1H, 0H\}$, each having probability $0.5 = \frac{1}{2}$.

Rolling a fair die is another interesting consideration:

It leads to 6 ELOs, namely, $\{1, 2, 3, 4, 5, 6\}$ (the possible number of dots on top), each having probability $\frac{1}{6}$ (roughly 0.1667). Note that $P(1) + P(2) + P(3) + P(4) + P(5) + P(4) = 1$.

Fair Coins and Dice

What if we toss 2 fair coins? (or the same coin twice, which ammounts to the same thing). If our focus is on counting Heads, we might list these possible outcomes: $\{2H, 1H, 0H\}$. Then a key question arises: are these 3 ELOs, each having probability $\frac{1}{3}$? No!

What about rolling 2 fair dice?

Let's focus on the total number of dots visible on top. Since each die comes up a number ranging from 1 to 6, the total is a number between 2 and 12. So the list of possbilities is $\{2, 3, 4, \ldots, 12\}$. How many things are in that list? It's 11. Do we have a list of 11 ELOs, each having probability $\frac{1}{11}$? No!

What went wrong? We need smarter lists of outcomes!

When we toss 2 fair coins, say a nickel and a dime (or a nickel followed by a dime) we need to realise that instead of counting Heads we can list these 4 possible outcomes: $\{HH, HT, TH, TT\}$, where, for instance, HT means the nickel came up Heads and the dime came up Tails. That is very different from TH which is the nickel coming up Tails and the dime coming up Heads.

What is $P(HH)$? Well, if we believe that Heads come up 50% (or a half) of the time on average for each coin, then it's reasonable to conclude that both come up Heads 50% of 50% (which is 25%) of the time, on average. Of course that's the same as $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

The same applied to $P(HT)$. It's also $\frac{1}{4}$. Likewise, $P(TH) = \frac{1}{4}$ and $P(TT) = \frac{1}{4}$. So $\{HH, HT, TH, TT\}$ is a list of 4 ELOs.

Notice how we effectively multiplied probabilities here, this is legal because the coins are "physically independent" of each other.

Back to counting heads

If our focus is on counting Heads then the list of outcomes $\{2H, 1H, 0H\}$ needs to be revisited. Note $P(2H) = P(HH) = \frac{1}{4}$ and $P(0H) = P(TT) = \frac{1}{4}$. That leaves $P(1H) = P(HT \text{ or } TH)$. Since HT and TH each occur 25% of the time, on average, we get $P(HT \text{ or } TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. So $\{2H, 1H, 0H\}$ are not ELOs, but we figured out what each probability is.

Let's apply similar reasoning to the 2 dice situation. Instead of looking at the total number of dots on top, let's look at all of the possibilities. Since each die can come up in one of 6 ways, then there are $36 = 6 \times 6$ possibilities overall: (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), ..., (6,4), (6,5), (6,6).

These are 36 ELOs, since for any particular pair, say $(1, 3)$, we have $P((1, 3)) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ because the first die comes up 1 a sixth of the time, on average, and the second die comes up 3 a sixth of the time, on average. The following table is helpful:

The totals when rolling 2 fair dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Now let's refocus on the total number of dots T={2,3, ...,11,12}. Note $P(T=2)=P((1,1))=\frac{1}{36}$. However, a total of 3 can arise in 2 ways, so $P(T=3)=P((1,2) \text{ or } (2,1))=\frac{1}{36}+\frac{1}{36}=\frac{2}{36}$. Similarly, a total of 4 can arise in 3 ways, so $P(T=4)=P((1,3)$ or $(2,2) \text{ or } (3,1))=\frac{1}{36}+\frac{1}{36}+\frac{1}{36}=\frac{3}{36}$. In a similar way, we find, $P(T=6)=P((1,5),(2,4),(3,3),(4,2) \text{ or } (5,1))=\frac{5}{36}$.