Basics of Probability

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Smart list of outcomes–ELOs

When we toss 2 fair coins, say a nickel and a dime (or a nickel followed by a dime) we need to realise that even if we are only interested in the number of Heads seen we should start with this list of 4 possible outcomes: $\{HH, HT, TH, TT\}$.

What is $P(HH)$? Since Heads come up 50% (or a half) of the time on average for each coin, it's reasonable to conclude that both come up Heads 50% of 50% (which is 25%) of the time, on average. Note that $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

The same applied to $P(HT)$. It's also $\frac{1}{4}$. Likewise, $P(TH) = \frac{1}{4}$ and $P(TT) = \frac{1}{4}$. So $\{HH, HT, TH, TT\}$ is a list of 4 ELOs.

Notice how we effectively multiplied probabilities here, this is legal because the coins are "physically independent" of each other.

Back to counting heads

If our focus is on counting Heads then the list of outcomes $\{2H, 1H, 0H\}$ needs to be revisited. Note $P(2H) = P(HH) = \frac{1}{4}$ and $P(0H) = P(TT) = \frac{1}{4}$. That leaves $P(1H) = P(HT \text{ or } TH)$. Since HT and TH each occur 25% of the time, on average, we get $P(HT \text{ or } TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. So $\{2H, 1H, 0H\}$ are not ELOs, but we figured out what each probability is.

Let's apply similar reasoning to the 2 dice situation. Instead of looking at the total number of dots on top, let's look at all of the possibilities. Since each die can come up in one of 6 ways, then there are $36 = 6 \times 6$ possibilities overall: (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), ..., (6,4), (6,5), (6,6).

These are 36 ELOs, since for any particular pair, say $(1, 3)$, we have $P((1, 3)) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ because the first die comes up 1 a sixth of the time, on average, and the second die comes up 3 a sixth of the time, on average. The following table is helpful:

The totals when rolling 2 fair dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Now let's refocus on the total number of dots T={2,3, ...,11,12}. Note $P(T = 2) = P((1,1)) = \frac{1}{36}$. However, a total of 3 can arise in 2 ways, so $P(T = 3) = P((1,2) \text{ or } (2,1)) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36}$. Similarly, a total of 4 can arise in 3 ways, so $P(T = 4) = P((1,3)$ or $(2,2) \text{ or } (3,1)) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{3}{36}$. In a similar way, we find, $P(T = 6) = P((1,5),(2,4),(3,3),(4,2) \text{ or } (5,1)) = \frac{5}{36}$. Do more!

Probability given a list of ELOs

There is a pattern common to the 4 examples we've looked at: 1 and 2 coin tosses, and 1 and 2 dice rolls. In each case we managed to come up with a list S of ELOs ("the sample space") so that for anything we wanted the probability of, we could use this formula:

$$P(E) = \frac{\#(E)}{\#(S)}$$

Here E is called "an event": it's the specific outcome of interest.

For instance, when rolling 1 die, we have $S = \{1, 2, 3, 4, 5, 6\}$ and if $E = \text{``}get\ 4''$ then $P(E) = \frac{1}{6}$.

If $E = \text{``get an even number''} = \{2, 4, 6\}$ then $P(E) = \frac{3}{6}$.

If $E = \text{``don't get 6''} = \{1, 2, 3, 4, 5\}$ then $P(E) = \frac{5}{6}$.

Note that $P(\text{don't get 6}) = 1 - P(\text{get 6})$.

Probability of something NOT happening

If the weather forecast says there is a 30% chance of rain, then of course there is a 70% chance of no rain.

We could write $P(\text{no rain}) = 1 - P(\text{rain}) = 1 - 0.30 = 0.70$.

This works in general:

$$P(\text{not } E) = 1 - P(E)$$

This can also be written as $P(E) = 1 - P(\text{not } E)$.

It works because

$$P(\text{not } E) = \frac{\#(\text{not } E)}{\#(S)} = \frac{\#(S) - \#(E)}{\#(S)}$$

$$= \frac{\#(S)}{\#(S)} - \frac{\#(E)}{\#(S)} = 1 - \frac{(\#E)}{(\#S)} = 1 - P(E)$$

Probability of something NOT happening

For a coin toss, we knew $P(Tail) = P(\text{not } Head) = 1 - P(Head)$.

Now consider 2 fair coin tosses. Recall $S = \{HH, HT, TH, TT)\}$ with 4 ELOs. We already know that $P(\text{exactly one Head}) = \frac{\#(HT,TH)}{4} = \frac{2}{4}$.

Find $P(\text{at least one Head}) = 1 - P(\text{NOT at least one Head}) = 1 - P(\text{NO Heads}) = 1 - P(TT) = 1 - \frac{1}{4} = \frac{3}{4}$. Of course it's also $\frac{\#(\text{at least one Head})}{4} = \frac{\#(HT,TH,TT)}{4} = \frac{3}{4}$.

Let T be the total for 2 fair dice. What's $P(T \text{ is not } 12)$? It's $1 - P(T = 12) = 1 - \frac{1}{36} = \frac{35}{36}$. What about $P(T \leq 10)$? It's $1 - P(T > 10) = 1 - P(T = 11 \text{ or } 12) = 1 - (\frac{2}{36} + \frac{1}{36}) = 1 - \frac{3}{36} = \frac{33}{36}$.

3 Fair Coin Tosses

Let's now consider 3 fair coin tosses. E.g., perhaps we toss 3 fair coins at the same time, say a nickel, a dime and a quarter, or we toss a single coin 3 times in a row. Even if we are only interested in the number of Heads seen we should start with a list of possible outcomes that we can show are ELOs.

Here's a list $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ of 8 possible outcomes. What is $P(HHH)$? Since Heads come up 50% of the time on average for each coin, it's reasonable to conclude that all 3 come up Heads 50% of 50% of 50% (which is 12.5%) of the time, on average. Note that $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

The same applied to $P(HHT)$. It's also $\frac{1}{8}$. Likewise, $P(HTH) = \frac{1}{8}$ and $P(HTT) = \frac{1}{8}$, and it's also true for the other 4 possibilities. So $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ is a list of 8 ELOs.

3 Fair Coin Tosses

Suppose with 3 fair coin tosses we are only interested in the total number of Tails seen. Then we get a new sample space $\{3T, 2T, 1T, 0T\}$. Are these 4 ELOs? Of course not!

Recall ELOs $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

Certainly $P(3T) = P(TTT) = \frac{1}{8}$ and $P(0T) = P(HHH) = \frac{1}{8}$.

Next, $P(2T) = P(HTT \text{ or } THT \text{ or } TTH)$. Since HTT and THT and TTH each occur $1/8$ of the time, on average, we get $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$.

Similarly $P(1T) = P(HHT \text{ or } HTH \text{ or } THH) = \frac{3}{8}$.

So $\{3T, 2T, 1T, 0T\}$ has 4 non-ELOs, but for each event in this list we found its probability, getting 4 numbers that add up to 1.