“At Least One” Problems

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"At Least One" Problems

When we toss 6 fair coins, there are many, many possibilities, too many to list and analyze in detail using what we've learned so far. If we are only interested in the number of Heads it's even harder.

But we could start with a list of possible outcomes:

$$\{HHHHHH, HHHHHT, HHHHTH, \ldots, TTTTTT\}$$

There are $2^6 = 64$ items on that list, and it turns out (once again) they are ELOs!

What is $P(\text{get no Heads})$? It's $P(TTTTTT) = (\frac{1}{2})^6 = \frac{1}{64}$. What is $P(\text{get at least one Head})$?

It's $1 - P(\text{get no Heads}) = 1 - \frac{1}{64} = \frac{63}{64}$. Repeat for 10 fair coins.

This technique works in other situations not based on ELOs.

Back to Babies

Consider births of new born babies. Instead of focussing on their weights or when they arrived, let's consider gender.

We're going to make two simplistic assumptions: (1) Each is either Girl or Boy and (2) Each of those options is equally likely.

(Google "Human Sex Ratio" for information on the more complex reality. The associated Wikipedia page is fuzzy about the fact that the ratio is different for births versus the general population versus those over 80.)

So we've replaced "tossing a fair coin" by "having a baby" or "picking a baby at random" where Girls occur 50% of the time, on average, and Boys the other 50% of the time. We don't expect that out of every 100 newborn babies exactly 50 are Girls. But while 4 Girls out of 10 babies is not unreasonable, 4 million Girls out of 10 million babies seems unlikely.

We write $P(G) = 0.5 = \frac{1}{2}$ and say "the probability of a Girl is 0.5". Similarly we write $P(B) = 0.5 = \frac{1}{2}$ and say "the probability of a Boy is 0.5". Note that $P(G) + P(B) = 1$. We have 2 **Equally Likely Outcomes** (ELOs), namely, $\{G, B\}$, each of which has probability $0.5 = \frac{1}{2}$. If our focus is on counting Girls, we get the equivalent list $\{1G, 0G\}$, each having probability $0.5 = \frac{1}{2}$.

What if we consider 2 babies? If our focus is on counting Girls, we might list these possible outcomes: $\{2G, 1G, 0G\}$. Just like for 2 coin tosses, these are not ELOs.

Before counting Girls we should list these 4 possible outcomes: $\{GG, GB, BG, BB\}$, where, for instance, GB means the first baby was a Girl and the second was a Boy. That is different from BG.

What is $P(GG)$? Well, if we believe that Girls come up 50% (or a half) of the time on average for each baby, then it's reasonable to conclude that both babies are Girls 50% of 50% (which is 25%) of the time, on average. Of course that's the same as $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

The same applies to $P(GB)$. It's also $\frac{1}{4}$. Likewise, $P(BG) = \frac{1}{4}$ and $P(BB) = \frac{1}{4}$. So $\{GG, GB, BG, BB\}$ is a list of 4 ELOs.

If our focus is on counting Girls then the list of outcomes $\{2G, 1G, 0G\}$ needs to be revisited. Note $P(2G) = P(HG) = \frac{1}{4}$ and $P(0G) = P(bb) = \frac{1}{4}$. That leaves $P(1G) = P(BT \text{ or } BG)$. Since GB and BG each occur 25% of the time, on average, we get $P(GB \text{ or } BG) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. So $\{2G, 1G, 0G\}$ are not ELOs, but we figured out what each probability is.

Another way of interpretting our discussions is to assert that:

50% of one child families (on average) have a Girl.

25% of two child families (on average) have two Girls.

50% of two child families (on average) have one Girl and one Boy.

25% of two child families (on average) have two Boys.

Probability of something NOT happening

For a single baby, we knew $P(Boy) = P(\text{not } Girl) = 1 - P(Girl)$.

Now consider 2 babies. Recall $S = \{GG, GB, BG, BB)\}$ with 4 ELOs. We have $P(\text{exactly one Girl}) = \frac{\#(GB, BG)}{4} = \frac{2}{4}$.

Find $P(\text{at least one Girl}) = 1 - P(\text{NOT at least one Girl}) = 1 - P(\text{NO Girls}) = 1 - P(BB) = 1 - \frac{1}{4} = \frac{3}{4}$. Of course it's also $\frac{\#(\text{at least one Girl})}{4} = \frac{\#(GG, GB, BG)}{4} = \frac{3}{4}$.

Similarly, $P(\text{at least one Boy}) = \frac{\#(GB, BG, BB)}{4} = \frac{3}{4}$.

3 Babies

Let's now consider 3 babies. Even if we are only interested in the number of Girls we should start with a list of possible outcomes that we can show are ELOs.

Here's a list $\{GHG, GGB, GBG, GBB, BGG, BGB, BBG, BBB\}$ of 8 possible outcomes. What is $P(GGG)$? Since Girls come up 50% of the time on average for each baby, it's reasonable to conclude that all 3 come up Girls 50% of 50% of 50% (which is 12.5%) of the time, on average. Note that $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

The same applies to $P(GGB)$. It's also $\frac{1}{8}$. Likewise, $P(GBG) = \frac{1}{8}$ and $P(GBB) = \frac{1}{8}$, and it's also true for the other 4 possibilities. So $\{GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB\}$ is a list of 8 ELOs.

3 Babies – How many Boys?

Suppose with 3 babies we are only interested in the total number of Boys. Then we get a new sample space $\{3B, 2B, 1B, 0B\}$. Are these 4 ELOs? Of course not!

Recall ELOs $\{GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB\}$.

Certainly $P(3B) = P(BBB) = \frac{1}{8}$ and $P(0B) = P(GGG) = \frac{1}{8}$.

Next, $P(2B) = P(GBB \text{ or } BGB \text{ or } BBG)$. Since GBB and BGB and BBG each occur $1/8$ of the time, on average, we get $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$.

Similarly $P(1B) = P(GGB \text{ or } GBG \text{ or } BGG) = \frac{3}{8}$.

So $\{3B, 2B, 1B, 0B\}$ has 4 non-ELOs, but for each event in this list we found its probability, getting 4 numbers that add up to 1.

3 Child Families

Another way of interpretting our discussions above is to assert that:

12.5% of three child families (on average) have three Boys.

37.5% of three child families (on average) have one Girl and two Boys.

37.5% of three child families (on average) have two Girls and one Boy.

12.5% of three child families (on average) have three Girls.

More Serious Applications

1. Now suppose you date 8 people over the course of your college career, and that 11% of the date pool are left-handed people. We'll assume that the other 89% are right-handed.

$P(\text{date at least one Lefty}) = 1 - \text{P(you date all Righties)} =$
$1 - \text{P(RRRRRRRR)} = 1 - (0.89)^8 = 1 - 0.3937 = 0.6063.$
That's an almost 61% chance of some "Lefty dating" experience.

2. Now suppose you run into 50 people over the course of a week in April 2020. Assume that 4% of them are carriers of a virus. We'll assume that the other 96% are healthy (denoted by H).

What is $P(\text{you meet at least one person with the virus})$?

It's $1 - P(\text{you meet none with virus}) = 1 - \text{P(HHHH} \ldots \text{H)} =$
$1 - (0.96)^{50} = 1 - 0.1300 = 0.8700$. That's an 87% chance of some exposure. Now try similar problems with different numbers.