“At Least One” and Birthday Problems

Colm Mulcahy

Math 107-03, Spring 2020, Spelman College

29 Apr 2020

"At Least One" Problems for Coins

When we toss 6 fair coins, there are many, many possibilities, too many to list and analyze in detail using what we've learned so far. If we are only interested in the number of Heads it's even harder.

But we could start with a list of possible outcomes:

$$\{HHHHHH, HHHHHT, HHHHTH, \dots, TTTTTT\}$$

There are $2^6 = 64$ items on that list, and they are ELOs!

What is $P(\text{get no Heads})$? It's $P(TTTTTT) = (\frac{1}{2})^6 = \frac{1}{64}$.

What is $P(\text{get at least one Head})$?

It's $1 - P(\text{get no Heads}) = 1 - \frac{1}{64} = \frac{63}{64}$.

Repeat for 10 fair coins.

“At Least One” Problems for Coins & Babies

When we toss 10 fair coins, there are $2^{10} = 1024$ possibilities, namely,

$$\{HHHHHHHHHH, HHHHHHHHHT, \ldots, TTTTTTTTTT\}$$

These are ELOs!

So $P(\text{get no Tails}) = \text{P(HHHHHHHHHH)} = (\frac{1}{2})^{10} = \frac{1}{1024}$.

What is $P(\text{get at least one Tail})$?

It's $1 - P(\text{get no Tails}) = 1 - \text{P(HHHHHHHHHH)} = 1 - (\frac{1}{2})^{10} = 1 - \frac{1}{1024} = \frac{1023}{1024}$.

Now consider 10 babies. What is $P(\text{get at least one Girl})$?

It's $1 - P(\text{get no Girls}) = 1 - \text{P(BBBBBBBBBB)} = 1 - (\frac{1}{2})^{10} = 1 - \frac{1}{1024} = \frac{1023}{1024}$.

"At Least One" Problems for Dice

Let's focus on the outcome "getting all 6's" when we roll dice.

When we roll 1 fair die, there are 6 ELOS, and $P((6)) = \frac{1}{6}$.

When we roll 2 fair dice, there are 36 ELOS, and $P((6,6)) = \frac{1}{36}$.

Suppose we toll 4 fair dice. There are $6^4 = 1296$ ELOS, and $P((6,6,6,6)) = \frac{1}{1296}$.

Let's switch focus to the outcome "getting no 6's", which means getting "all non-6's" (namely, all dice came up with some number between 1 and 5). Let N denote "Not 6" on one partictular die.

When we roll one die $(\mathrm{no}\ 6) = \mathrm{P(N)} = \frac{5}{6} = 0.8333$.

When we roll two dice $(\mathrm{no}\ 6) = \mathrm{P(NN)} = (\frac{5}{6})^2 = 0.6944$.

When we roll four dice $(\mathrm{no}\ 6) = \mathrm{P(NNNN)} = (\frac{5}{6})^4 = 0.4823$.

"At Least One" Problems for Dice

Now, focus on the outcome "getting at least one 6" when we roll dice. What's the probability that this occurs?

For one die it's $P(\text{get at least one 6}) = 1 - \text{P(get no 6)} = 1 - \text{P(N)} = 1 - \frac{5}{6} = 1 - 0.8333 = 0.1667.$

For two dice it's $P(\text{get at least one 6}) = 1 - \text{P(get no 6)} = 1 - \text{P(NN)} = 1 - (\frac{5}{6})^2 = 1 - 0.6944 = 0.3056.$

For four dice it's $P(\text{get at least one 6}) = 1 - \text{P(get no 6)} = 1 - \text{P(NNNN)} = 1 - (\frac{5}{6})^4 = 1 - 0.4823 = 0.5177.$ So for four dice, it's slightly more likely to happen than not happen.

This technique works in other situations not based on ELOs.

More Serious Applications

1. Now suppose you date 8 people over the course of your college career, and that 11% of the date pool are left-handed people. We'll assume that the other 89% are right-handed.

$P(\text{date at least one Lefty}) = 1 - \text{P(you date all Righties)} = 1 - \text{P(RRRRRRRR)} = 1 - (0.89)^8 = 1 - 0.3937 = 0.6063.$ That's an almost 61% chance of some "Lefty dating" experience.

2. Now suppose you run into 50 people over the course of a week in April 2020. Assume that 4% of them are carriers of a virus. We'll assume that the other 96% are healthy (denoted by H).

What is $P(\text{you meet at least one person with the virus})$?

It's $1 - P(\text{you meet none with virus}) = 1 - \text{P(HHHH} \ldots \text{H)} = 1 - (0.96)^{50} = 1 - 0.1300 = 0.8700.$ That's an 87% chance of some exposure. Now try similar problems with different numbers.

Birthday Problems

Here's a famous Dilbert cartoon:

Why is it funny?

Birthday Problems

First, we focus on days of the week. Pick a person at random, there are 7 days of the week they could have been born on:

$$\{Mon, Tue, Wed, Thu, Fri, Sat, Sun\}$$

We're going to assume these are ELOs, just to simplify the mathematics. We know it's not the case in real life (why not?).

What is the probability that one randomly selected person was born on a Sat or Sun? It's $P(\text{Sat or Sun}) = \frac{2}{7}$.

What is the probability that one randomly selected person was not born on a Mon? It's $1 - P(\text{Mon}) = 1 - \frac{1}{7} = \frac{6}{7}$.

What is the probability that one randomly selected person *born on a weekday* was born on a Mon or a Fri? It's $\frac{2}{5} = 0.40$.

Birthday Problems

Now pick 2 people at random. This gives rise $7^2 = 49$ ELOs: $\{$(Mon,Mon), (Mon,Tue), ..., (Mon,Sun), ..., (Sun,Sun)$\}$.

What's prob that both people were born on a Wed? It's $(\frac{1}{7})^2 = \frac{1}{49}$.

What' prob that both people were born on a Sat or Sun? It's $(\frac{2}{7})^2 = \frac{4}{49}$.

What is the probability that neither of them were born on a Tue? It's $P(\text{both born on a non Tuesday}) = (\frac{6}{7})^2 = \frac{36}{49}$. So the prob that at least one of them was born on a Tue is $1 - \frac{36}{49} = 0.2653$.

The prob that they were born on different days of the week? It's $\frac{7}{7} \times \frac{6}{7} = \frac{6}{7} = 0.8571$.

Hence, the prob that they were born on the same day of the week is $1 - P(\text{they were born on diff days}) = 1 - \frac{6}{7} = \frac{1}{7} = 0.1429$.

Birthday Problems

Now pick 5 people at random. This gives rise to a sample space of $7^5 = 16807$ ELOs which we are not going to hint at a list of.

What's prob that all of them were born on a Wed? It's $(\frac{1}{7})^5$.

What's prob that all of them were born on a Sat or Sun? It's $(\frac{2}{7})^5$.

What's prob that none of them were born on a Tue? It's $P(\text{all born on a non Tuesday}) = (\frac{6}{7})^5 = 0.4647$. So prob at least one of them was born on a Tue is $1 - 0.4647 = 0.5373$.

What is the probability that they were all born on different days of the week? It's $\frac{7}{7} \times \frac{6}{7} \times \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = 0.1499$. Hence, the prob that at least two of them were born on the same day of the week is $1 - P(\text{they were born on diff days}) = 1 - 0.1499 = 0.8501$.

Moral: coincidences shouldn't surprise us! Some of them are very likely to occur.

Real Birthday Problems

Now pick 10 people at random. Let's assume each person's birthday is considered as a day of the year, assuming there are 365 choices ranging from 1 Jan to 31 Dec (we ignore leap years).

What's prob that all of them were born on 4 Jul? It's $(\frac{1}{365})^{10}$.

What's prob that none of them were born on 25 Dec? It's $P(\text{all born on non Xmas}) = (\frac{364}{365})^{10} = 0.9729$. So prob at least one of them was born on 25 Dec is $1 - 0.9729 = 0.0271$.

What's prob that they were all born on different days of the year? It's $\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{356}{365} = 0.8831$.

Hence, prob that at least two of them were born on the same day of the year is $1 - P(\text{all diff days}) = 1 - 0.8831 = 0.1169$.

It's not to surspiring that with 10 people we don't expect some birthday coincidence. So let's increase the number of people.

The Birthday Paradox

Consider 23 randomly selected people. Repeating the kinds of calculations just studied shows that the probability that at least two of them were born on the same day of the year is 0.5005. That's slightly over 50%. Most people find this extremely surprising and it's known as The Birthday Paradox.

For 30 people it's 69.68%. For 50 people it's 96.63%.

For 100 people it's very close to 100%.

Google "birthday paradox calculator" to get websites where these numbers can easily be confirmed.

You *can* find 100 people with different birthdays, you can even find 300 (or more) such people. But you have to try very hard.

(You cannot, however, find 400 people with different birthdays!)

Coincidences shouldn't surprise us! Some are very likely to occur.