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Fitch Cheney's Five Card Trick

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In recent years a classic mid-20th century two-person card trick has resurfaced in the mathematical community. It goes something like this: A volunteer chooses five cards at random from a standard deck, hands them to you, and you show four of them to your confederate who promptly names the fifth card.

This superb effect is usually credited to mathematician William Fitch Cheney, Jr. The original trick, and some generalizations discussed here, work as well with large audiences as with small ones. The tricks are 100% mathematical—though you may choose to dress them up a little, for instance as "mind-reading tricks." They will stump all but the most sophisticated onlookers; a general audience will be convinced that some sort of body language or verbal signaling is being used. To eliminate that possibility, one can utilize email, telephone, an innocent go-between, or some other form of "impersonal" communication. Indeed, Cheney's trick, as published by W. Wallace Lee in 1950, was intended to be carried out over the telephone.

Cheney's Five-Card Trick

Effect: A volunteer from the crowd chooses any five cards at random from a deck, and hands them to you so that nobody else can see them. You glance at them briefly, and hand one card back, which the volunteer then places face down on the table to one side. You quickly place the remaining four cards face up on the table, in a row from left to right. (You can put them in a pile face down if you are worried that the audience thinks there is some subtle signaling going on in the precise placement of the cards on the table!) Your confederate, who has not been privy to any of the proceedings so far, arrives on the scene (e.g., is called in from another room), inspects the faces of the four cards, and promptly names the hidden fifth card.

Method & Mathematics: We strongly recommend that you spend some time (possibly days!) thinking over just how this puzzling trick might work before you read any further. It will give you great pleasure if you work out a solution for yourself. Yes, it really does work given any five cards.

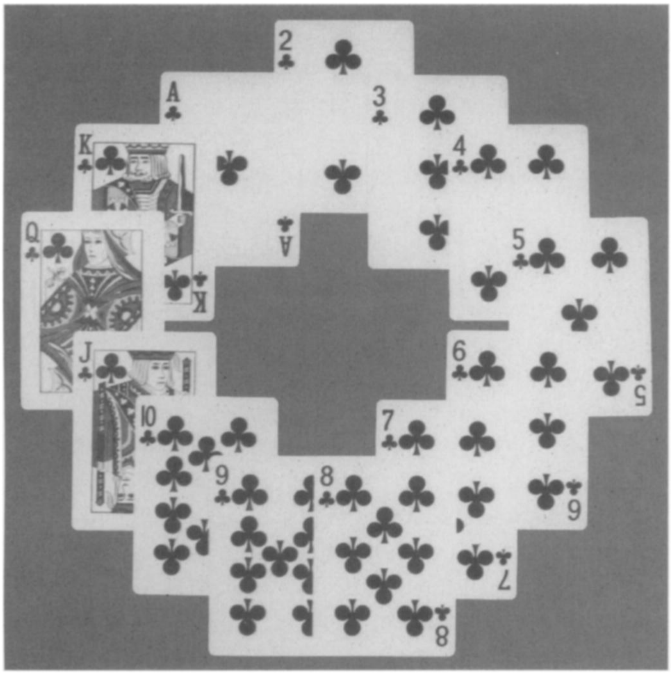
Are you sure you are ready to read on? There are three distinct parts to the solution we present here.

Note that *you* get to choose which card to hand back (and later on, in what order to place the remaining four cards). First, the pigeonhole principle guarantees that (at least) two of the five cards are of the same suit. Let's suppose that you have two Clubs. One Club is handed back, and by placing the remaining four cards in some particular order, you effectively tell your confederate the identity of the Club you just handed back.

The second main idea is this: you can use one designated position (e.g., the first) of the four available on the table for the retained Club—which *determines the suit of the hidden card*—and the other three for the placement of the remaining cards, which can be arranged in $3! = 6$ ways. If you and your confederate agree in advance on a one-to-one correspondence between the six possible permutations and $1, 2, \ldots, 6$, then you can communicate one of six things.

What can one say about these other three cards? Not much—for instance, some or all of them could be Clubs, or there could be other suit matches! However, one thing is certain: they are all distinct, so with respect to some total ordering of the entire deck, one of them is LOW, one is MEDIUM, and one is HIGH. This permits for an unambiguous and easily remembered way to communicate a number between 1 and 6. But surely six isn't enough? The hidden card could in general be any one of twelve Clubs!

This brings us to the third main idea: you must be careful as to exactly which Club you hand back! Considering the 13 possible card values, 1 (Ace), 2, 3, . . ., 10, J, Q, K, to be arranged clockwise on a circle, we can see that the two Clubs picked are at most 6 values apart, i.e., counting clockwise one of them lies at most 6 vertices past the other. Give this "higher" valued Club back to the victim, which they then hide. You'll use the

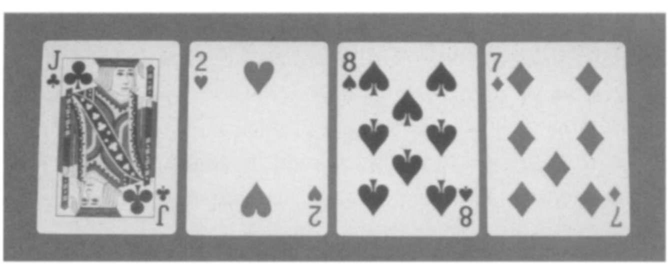


Counting clockwise, any two clubs are no more than six places apart.

"lower" Club and the other three cards to communicate the identity of this hidden card to your confederate.

For example, if you have the 2♣ and 8♣, then hand back the 8♣, but if you have the 2♣ and J♣, hand back the 2♣. In general, you save one card of a particular suit and need to communicate another of the same suit, whose numerical value is k higher than the one you make available, for some integer k between 1 and 6 inclusive.

Put this total linear ordering on the whole deck: A♣, 2♣, . . ., K♣, A♦, 2♦, . . ., K♦, A♥, . . ., K♥, A♠, . . ., K♠. Mentally label the three cards L (low), M (medium), and H (high) with respect to this ordering. Order the six permutations of L, M, H by rank, i.e., 1 = LMH, 2 = LHM, 3 = MLH, 4 = MHL, 5 = HLM and 6 = HML. Finally, order the three cards in the pile from left to right according to this scheme to communicate the integer desired. (See Trick 1.)



Trick 1. The jack communicates the suit, the other three cards communicate the number four. Hence the hidden card is the two of clubs.

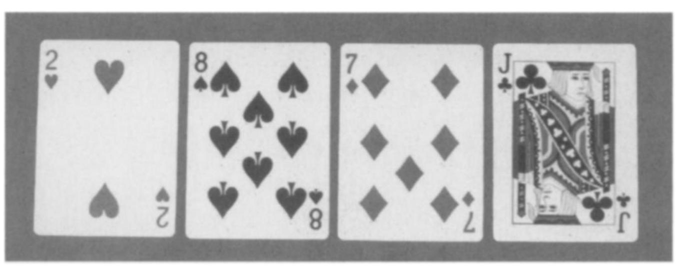
The one weakness in the method as described above is the invariant use of the first position in the pile as the "suit giver," this is soon spotted by alert audiences if the trick is repeated. Here is a better idea: since both you and your confederate get to see the four cards in the pile, sum their values and reduce mod 4 (using 4 if you get 0), and use that number for the position in the pile of the suit-determining card. For example, a Jack, 8, 2 and 7 would result in $11 + 8 + 2 + 7 = 0 \pmod 4$, so the fourth position in the pile would be used for the suit-determining card, and the first three cards would tell your confederate what to add to the numerical value of the last card to get the hidden card.

To return to the earlier example, suppose the five cards you are handed are 2♣, 2♥ 8♠, 7♦, and J♣. You play the J♣, in the 4th position, communicate $k = 4$ (hence the 2♣) to your confederate using the other three cards as follows: In standard LMH order they are 7♦, 2♥, 8♠, so in MHL order they are 2♥, 8♠, 7♦. (See Tricks 2 and 3.)

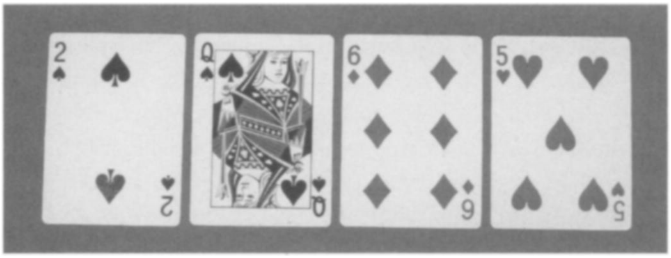
Of the three key ingredients in this trick, the pigeonhole principle and the permutations idea are easier to stumble upon than the fact one has only to communicate one of six (as opposed to twelve) integers.

Source: The trick is at least fifty years old, and seems to have first appeared under the name *Telephone Stud* in *Math Miracles* by W. Wallace Lee, where it is attributed to William Fitch Cheney, Jr, Chairman, Department of Mathematics, University of Hartford, Hartford, CT. Thanks to Art Benjamin for providing this source, and to Paul Zorn for alerting us to the existence of the trick in the first place. Gardner mentions it in passing in his *Mathematics, Magic and Mystery* as well as in his *The Unexpected Hanging and Other Mathematical Diversions*. This article was inspired by Brain Epstein's article "All You Need is Cards," published by A.K. Peters in the January 2002 *A Puzzlers' Tribute—A Feast for the Mind*, a tribute to Martin Gardner.

The basic Cheney trick is spelled out eloquently and concisely in J.H. van Lint and R. M. Wilson's *A Course in Combinatorics, 2nd Edition*. A much watered-down version, in which all five cards are assumed to be of the same suit, and non-mathematical signaling also takes place, appears in Karl Fulves's *More Self-Working Card Tricks*, with a nod to Cheney.



Trick 2. Again the jack communicates the suit and the other cards communicate the number four; the hidden card is the two of clubs.



Trick 3. What is the hidden card? (Answers are on page 13.)

More recently, it has been noted that this trick generalizes to larger decks, of up to 124 cards. See, for example, the January 2001 issue of *Emissary* or Michael Kleber's article in *The Mathematical Intelligencer,* Winter 2002. See also "Using a Card Trick to Teach Discrete Mathematics," Shai Simonson and Tara Holm, to appear in *PRIMUS*, 2003.

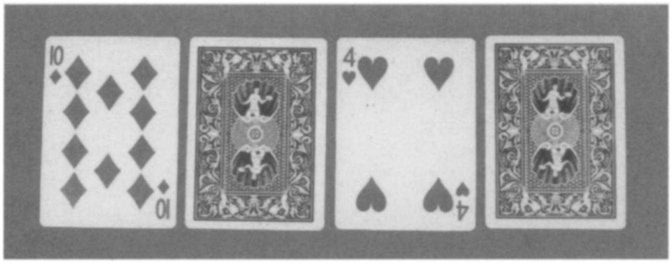
We now present some variations on Cheney's trick.

Ups and Downs and Back Again

Effect: A volunteer from the crowd chooses any five cards at random from a deck, and hands them to you so that nobody else can see them. You glance at them briefly, and hand one card back, which the volunteer then places face down on the table to one side. You quickly place the remaining four cards in a row on the table, some face up, some face down, from left to right. Your confederate, who has not been privy to any of the proceedings so far, arrives on the scene, looks at the cards on display, and promptly names the hidden fifth card.

Method & Mathematics: Again, you may wish to think this over before reading on.

As before, there are three distinct ideas at work here, two of which are the same as for the last trick. The pigeonhole principle guarantees that (at least) two of the five cards are of the same suit, without loss of generality you have two Clubs. You hand one back, and by placing the remaining four cards in some particular arrangement you effectively tell your confederate the identity of the hidden card. Save the "lower" Club, and communicate the identity of the "higher" one, whose numerical value is k past the one you hold on to, for some integer k between 1 and 6 inclusive. Use one particular position (e.g., the first) of the four available for the retained Club—which determines the suit of the hidden card—and the other three for the remaining three cards.



Trick 4. What is the hidden card?

The difference here is that you communicate k using some kind of binary code—rather than permutations—for the three free slots. Unlike in the last trick, the *identities* of the face up cards (in the three relevant slots) play no role here! Rather than use actual binary representations, let's agree on this convention: UDD, DUD, DDU (only 1st, 2nd or 3rd position is Up), and DUU, UDU, UUD (only 1st, 2nd or 3rd position is Down), respectively, reveal to your confederate which of 1, 2, 3 and 4, 5, 6 the integer k is. (Try Trick 4.)

Since *UUU* is avoided above, an audience for whom you seem to be repeating the original Cheney effect can be given a choice of which trick they wish to see, even in mid-play—while your confederate is out of the room—without ever realizing that they are deciding between two different tricks! One could, for instance, say, "Should we make it harder on her this time, and only show her *some* of the cards?" Regardless of the answer, as soon as your confederate sees the cards, she immediately knows exactly which trick she is doing.

If some cards are face down she can play up the impossibility of the task before her, and the unfairness of it all, before dumbfounding the crowd by correctly announcing the identity of the hidden card. One more piece of drama can be added to this trick. Since the DDD possibility for card placement is also avoided above, *at least one card of these three is always face up*. Hence, we can use a face-up card (let's agree on the first such if there are two) to communicate the suit.

When all is said and done, this means that only three cards of the four retained need to be shown at all (face up or face down), the fourth can be set aside and ignored, every time! This too can be used to spice up the trick upon repeat performances.

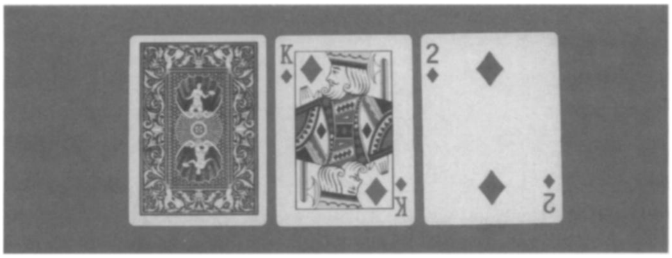
Source: Original.

A similar trick can be done with fewer cards.

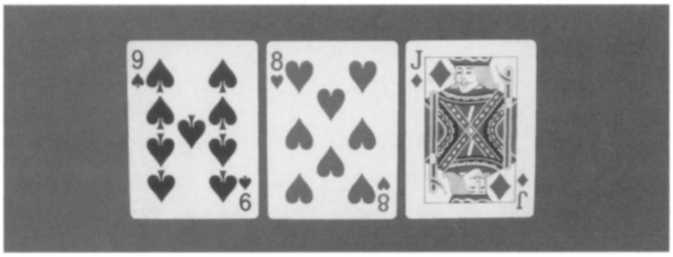
The Four-Card Cheney Trick

Effect: A volunteer from the crowd chooses any four cards at random from a deck, and hands them to you so that nobody else can see them. You glance at them briefly, and hand one back, which the volunteer then places face down on the table to one side. You quickly place the remaining three cards in a row on the table, some face up, some face down, from left to right. Your confederate, who has not been privy to any of the proceedings so far, arrives on the scene, looks at the cards on display, and promptly names the hidden fourth card—even in the case where all three cards are face down!

Method & Mathematics: We partition the standard deck into three new suits of 17 cards each, leaving out one special card (say A♠). Each of these new suits consists of the one of the standard suits ♣, ♦, ♥ supplemented with four ♠'s. Specifically, Suit A is A♣, 2♣, . . ., K♣, 2♠, 3♠, 4♠, 5♠; Suit B is



Trick 5. What is the hidden card?



Trick 6. What is the hidden card?

A♦, 2♦, . . ., K♦, 6♠, 7♠, 8♠, 9♠; and Suit C is A♥, 2♥, . . ., K♥, 10♠, J♠, Q♠, K♠.

If one of the four cards is the special card, A♠, play the other three face down and it's all over: your confederate can identify the last card with what appears to be zero information! Otherwise, the pigeonhole principle guarantees that (at least) two of the four cards are from one of the three redefined suits, without loss of generality Suit A. You hand one back, and then by placing the remaining three cards on the table in some particular fashion you reveal to your confederate the identity of the card handed back.

As before, save the "lower" card from Suit A, and communicate the "higher" one, whose numerical value is k past the one you hold on to, where this time k is an integer between 1 and 8 inclusive. In the convention we explain below, at least one card will be face up, so once more we can use a face up card (the first such if there are two) to communicate the suit. As suggested before, the placements *UDD, DUD, DDU* (one *U* in 1st, 2nd or 3rd position) and *DUU, UDU, UUD* (one *D* in 1st, 2nd or 3rd position), respectively, can be used to tell your confederate that k is 1, 2, 3, 4, 5 or 6. This time we also need a way to communicate 7 or 8, and we have the *UUU* option at our disposal. If we agree to use one particular *U* (say, the middle one) to give the suit, there are two ways to play the other two: Low-High (to convey $k = 7$) or High-Low (for $k = 8$) with respect to some total ordering of the deck, such as lining up Suits A, B, C in that order. (Try Tricks 5 and 6.)

Source: Original.

What do you do in the original five-card trick if the volunteer insists on chosing the card to be identified? Characteristically, Martin Gardner determined an out many years ago; however, be warned, it does require a certain amount of mental gymnastics.

Eigen's Value

Effect: A volunteer from the crowd chooses any five cards from a deck, and hands them to you so that nobody else can see them. The volunteer then takes one card of their choosing back, which they show around to everybody, before plunging it into the deck and shuffling thoroughly. You quickly place the remaining four cards in a row from left to right on the table. Your confederate arrives on the scene, inspects the cards on the table, as well as the remainder of the deck, and asks what the chosen card was. Upon it being named, the top card of the deck is turned over, and sure enough it is the chosen card.

Method & Mathematics: The mathematics is simple here, it's the method that requires some memory work. Unlike in the classic Cheney trick, you do not get to choose which card is the one your confederate must identify. So, in essence, you have to use the four cards you have to try to communicate which of the remaining $52 - 4 = 48$ possible cards it is. Since $4! = 24$, you can narrow it down to one of two cards by communicating a permutation. Here is one way: fix a total ordering on the deck as before. The four seen cards now determine two things: which 48 cards remain as possibilities for the hidden fifth card, and a particular permutation of the 24 available. Assuming that we rank those permutations as before, this means that a number, say 15, is communicated.

Your confederate must now mentally determine the 15th and 39th cards in the remaining ordered list of 48 possible cards: they are the cards that would usually be in those positions with respect to the agreed-upon total ordering, bumped up one for every seen card which occurs before those positions. The confederate locates these in the remainder of the deck, brings them together discreetly, and cuts the deck so as to bring one to the top and the other to the bottom. Finally, upon the naming of the chosen card, your confederate either turns over the top card, or turns over the deck to reveal the bottom card. This unpredictable last step makes the trick unsuitable for repeating, but as a one-off follow-up to the basic Cheney it is suitably mysterious.

Another way to get from 1 of 24 permutations to 1 of 48 cards is to provide one additional bit of information, for instance the displayed cards could be laid out right to left or left to right (if your confederate is allowed to see this part of the proceedings) or you could provide just one subtle physical or verbal signal.

Source: This is slightly adapted from Victor Eigen's trick as found in Gardner's *The Unexpected Hanging and Other Mathematical Diversions*. ■

Answers

Trick 3. 7♠. Trick 4. Q♦. Trick 5. 9♠. Trick 6. Q♠.