Martin Muldoon 1939-2019

Kathy A. Driver and Mourad E. H. Ismail

September 26, 2019

Filename: Muldoonobit25Sept.tex

Martin Muldoon was born on February 28, 1939 in Westport, County Mayo, Ireland. He was one of eight children of Peter and Margaret (Hopkins) Muldoon. He studied Mathematics at University College Galway, now National University of Ireland Galway, obtaining a BSc in 1959 and an MSc in 1960. Jim Muldowney, who was one year junior to Martin in Mathematics at University College Galway , recalls that he was a popular figure at college although modest and self-effacing. He was bright and funny and kind and was well-known for his gentle wit as a frequent speaker at "Lit and Deb", often a rowdy and daunting forum. In 1961 he went to the University of Alberta in Edmonton to pursue graduate studies in Mathematics leading to the PhD in 1966. Jim also joined the PhD program at the University of Alberta a year later and shared an apartment with Martin. They purchased a VW Beatle in which they spent a memorable camping holiday touring western Canada and north-western USA. In those early days of the Vietnam war, Martin was already well-known for his commitment to pacifism and social justice. His sunny demeanor and kindness made him a popular student on this ethnically and culturally diverse campus also.

After graduation from the University of Alberta, Martin joined the faculty at York University as an assistant professor and advanced through the ranks where he became a professor in 1978. He retired in 2004. Some information is in the York University obituary [10].

This paper is a tribute to a dear friend and long time collaborator of both authors. We will first describe how we met Martin and briefly mention out joint research, then we will discuss his other contributions. Chandler Davis kindly contributed the part about Martin's activities as part of the Science for Peace movement.

We met Martin at different stages of our careers. Mourad met him in the early 1970's possibly at a mathematical meeting in Edmonton but certainly at the 1974 CBMS conference where Askey was the principal lecturer. They met again in Toronto when Mourad was a post doctorate fellow at the University of Toronto with partial support from York University where Martin was a faculty member. Their first paper was on a problem proposed by Thomas Nagylaki, University of Chicago. The question was to determine the dependence of the smallest eigenvalue of a boundary value

problem on $\mathbb{R}^n$ on the dimension n. This amounted to showing that the roots of a transcendental equation are monotone in the dimension n. The key to the solution was to treated n as a continuous variable. With the new formulation they each had a solution and the paper contained both solutions. Martin's solution used the Hellmann-Feynman theorem and this was the first time Mourad ever heard about it. Roughly stated the Hellmann-Feynman theorem says that if $H(a)$, is a family of Hermitian operators with a in a real interval and $H(a)$, has an eigenfunction v, $\|v\| = 1$ corresponding to an eigenvlalue $\lambda(a)$ then under some conditions, [6],

$$\frac{\partial \lambda(a)}{\partial a} =< \left(\frac{\partial H(a)}{\partial a} \right) v, v > . \tag{1}$$

Note that the eigenvector, which we may not know explicitly, is not differentiated. A rigorous treatment of this amazing theorem is in [6] and was motivated by a lecture of Martin at Arizona State University, where the Hellmann-Feynman theorem and its applications were mentioned. Many applications are collected in [4].

In the late 1970's Mourad was interested in complete monotonicity of quotients of special functions and their relation to infinitely divisible distributions. Martin was also interested in the subject and they started to collaborate. They also wrote a paper with Lee Lorch on a related problem. Martin was also interested in bounds and monotonicity of of zeros of special functions and they wrote several papers on this topic. One of their deepest results is [5] ([M38]). They used chain sequences and the Hellmann-Feynman theorem to obtain sharp bounds on zeros of orthogonal polynomials. More importantly they provided a technique to study this problem in specific cases. They also showed how the concept of a chain sequence comes from the process of reduction of a positive definite matrix to its row echelon form. Many of their results are mentioned in [4]. The Ismail-Muldoon collaboration was a learning experience for both. Mourad learned the Hellmann-Feynman technique from Martin, and it is fair to say that Mourad infected Martin with the q-disease, [3]. As a result Martin became interested in the q-gamma function and zeros of q-polynomials [M19], [M28]. [33], [M42]. [M60], [M61], [M66].

Kathy met Martin late in his academic career (and late in hers) at the International Conference on Differential Equations, Difference Equations and Special Functions, Patras, Greece, in September 2012. Martin chaired her talk and, on the bus going on the excursion, they chatted about ideas and problems involving zeros of classical orthogonal polynomials. Plentiful emails followed and Martin's great talent for looking at properties of zeros of orthogonal polynomials using imaginative numerical plots conjured up using Maple emerged rather quickly. He taught Kathy a lot about the fascinating history of zeros of Bessel functions dating back to Bourget's Hypothesis and enjoyed pondering "obscure" results related to zeros of Bessel functions. They wrote 7 papers together and he was always a source of new ideas and fresh perspectives. In [1], Martin squeezed information out of non-trivial inequalities involving zeros of Laguerre polynomials and zeros of Bessel functions to prove sharp results for the interlacing properties of zeros of Laguerre polynomials of different degrees corresponding to different parameters. Without knowing Pálmai's result in [7] that the

positive zeros of the Bessel functions $J_\mu(x)$ and $J_\nu(x)$, $\mu, \nu > 0$ are interlacing if and only if $|\mu - \nu| \leq 2$, Martin exploited connections between zeros of Bessel functions and zeros of Laguerre polynomials that are used in their proof that for $\alpha > -1$, the zeros of the non-equal degree Laguerre polynomials $L_n^\alpha(x)$ and $L_{n-k}^{\alpha+t}(x)$ are interlacing for each $k \in 1, ..., n-1$ and every $n \in \mathbb{N}$ if and only if $0 < t \leq 2k$.

In the final paper that Martin and Kathy wrote together, which will appear soon in J. of Approximation Theory, Pálmai's result is used to to prove that for $\alpha > -1$, the zeros of the equal degree Laguerre polynomials $L_n^\alpha(x)$ and $L_n^{\alpha+t}(x)$ are interlacing for every $n \in \mathbb{N}$ and each fixed $\alpha > -1$ if and only if $0 < t \leq 2$.

Martin and Jay held generous, inclusive views of the world and he will be greatly missed by our community of mathematicians.

Martin was strongly influenced by his doctoral supervisor Lee Lorch, see the Wikipedia article [9] for abiography of Lee Lorch. Lorch and Muldoon were colleagues at York University from 1968 till Lorch's death in 2017. Martin's first paper, [M2] which was based on his thesis proves the compete monotonicity of the sequence

$$M_k := \int_{x_k}^{x_{k+1}} |y(x)|^\lambda dx, \tag{2}$$

where y is an oscillatory solution of

$$y'' + f(x)y(x), \tag{3}$$

$x_1 < x_2 < \cdots$ are the zeros of y, and $\lambda > -1$. This extended an earlier results of Lorch and Peter Szego. A related question is to study the complete monotonicity of $\rho(x) := y_1^2(x) + y_2^2(x)$, where y_1 and y_2 are linear independent solutions of the differential equation (3). Martin and his coauthors wrote several additional papers on the same theme. In some papers they considered a special form of f.

In his papers on singular integrals, one objective is to prove

$$\lim_{\nu \to \infty} \int f(t) w(\nu, t - x) dt = \frac{2}{3} f(x-) + \frac{1}{3} f(x+),$$

where w is a suitable normalized solution if

$$\frac{d^2 w}{dt^2} = [\nu^2 t + q(t)] w,$$

under various assumptions of q and f. The first paper on this topic, [M3], was based on a chapter in Martin's thesis. A theorem of R. G. Cook, [8], [2] states that the areas under successive arches of the graph of $J_\nu(x), \nu > -1$ form a decreasing sequence. In [M9] this extended to a general cylindrical function times t^{-a} for some range of a.

Martin worked on the gamma and q-gamma functions [3]. Characterizations of the gamma function are given in [M8] and [M13], while [M19] dealt with a characterization of the q-factorial function. Many inequalities, complete monotonicity of quotients of

products of gamma and q-gamma functions are in [M26], [M42], and [M66]. Another question Martin worked on was to characterize certain special functions by nonlinear equations satisfied by their zeros. This is a question originated in physics and first addressed by researchers from the Calogero team in Rome. This topic was explored in [M16] and [M23].

Martin provided valuable service to the SIAM activity group on orthogonal polynomials and special functions for many years. He was program director for 1993-95, he edited the activity group's newsletter from 1999-2007 and was editor from 1996 till 1998.

Martin wrote several papers using Sturmian arguments to analyze zeros of special functions and orthogonal polynomials. His survey article [M29] with Laforgia is very informative. He collaborated with many mathematicians on using Sturmian techniques to study monotonicity properties of special functions. Notable collaborators are Elbert, Laforgia, Lorch, and Spigler.

Martin was a collaborator of Lee Lorch in peace work as in mathematics. It was natural that when Lee died in 2014, Martin wrote a widely read obituary mentioning his work for peace and civil liberties. He also led in founding a fund at York University in honor of Lee, dedicated to these goals.

Both Lee and Martin were active over the years in Science for Peace, an activist organization since 1981 primarily of university people, headquartered in Toronto. Martin served on the Executive Committee for many years. He was one of the most reliable and unfailingly cheerful in a sometimes fractious lot. His fairness and even temper became especially important in a period of dissension within the organization over several years; when the dust settled, the leaders again were willing to work together, but some of the older group failed to put in the effort to restore constructive activity. Martin as Treasurer was one of a small nucleus that filled the gap and enabled Science for Peace to move forward. His death will be mourned by all, but his legacy will be a harmonious, competent peace group well able to carry on his spirit.

Martin had a very pleasant personality and it was always a pleasure to meet him, work with him, or just have a social visit. He and his wife Joy were gracious hosts and we always looked forward to see them.

The authors gratefully acknowledge the information provided by James Muldowney from the University of Alberta, about Martin's youth and college days and Chandler Davis from the University of Toronto about Martin's active participation in the Scientist for Peace movement.

References

[1] K A Driver and M E Muldoon. Common and interlacing zeros of families of Laguerre polynomials. Journal of Approximation Theory Vol. 193 (2015), 89-98.

[2] A. Erdélyi and W. Magnus and F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, volume 2, McGraw-Hill, New York, 1953.

[3] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, second edition Cambridge University Press, Cambridge, 2004.

[4] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in one variable*, Cambridge University Press, paperback edition, Cambridge, 2009.

[5] M. E. H. Ismail and M. E. Muldoon, A discrete approach to monotonicity of zeros of orthogonal polynomials, Trans. Amer. Math. Soc. **323** (1991), 65–78.

[6] M. E. H. Ismail and R. Zhang, On the Hellmann-Feynman theorem and the variation of zeros of special functions, Advances in Appl. Math 9 (1988), 439–446.

[7] T. Pálmai. On the interlacing of cylinder functions, Math. Inequal. Appl., **16** (2013), 241–247.

[8] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, second edition, Canbridge University Press, Cambridge, 1944.

[9] https://en.wikipedia.org/wiki/Lee_Lorch

[10] https://yfile.news.yorku.ca/2019/08/14/passings-martin-muldoon/

K. Driver, Department of Mathematics, University of Cape Town, Cape Town 7708, South Africa
email: Kathy.Driver@uct.ac.za

M. E. H. I, Department of Mathematics, University of Central Florida, Orlando, Florida 32816
email: ismail@math.ucf.edu