Martin-Luther-University, Halle

Mathemagic with a Deck of Cards

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We survey new and old principles in mathemagic and their application to card tricks. These provide an entertaining and often surprising forum for mathematics which can be used

(1) to generate student interest in the subject,

and

(2) as a jumping off point for undergraduate independent study and research.

Predictions, mindreading, unaddition, and more!

Learn new tricks to entertain family and friends.

Consider a Regular Deck of 52 Playing Cards

Note that $52 = 26 + 26 = 13 + 13 + 13 + 13$.

A deck has 26 black ($\clubsuit$ & $\spadesuit$) and 26 red ($\heartsuit$ & $\diamondsuit$) cards.

There are 13 of each of the 4 suits (note the CHaSeD order) :

$\clubsuit$ (Clubs), $\heartsuit$ (Hearts), $\spadesuit$ (Spades), $\diamondsuit$ (Diamonds)

A$\clubsuit$ (Ace), 2$\clubsuit$,. . . , 10$\clubsuit$, J$\clubsuit$ (Jack), Q$\clubsuit$ (Queen), K$\clubsuit$ (King),

A$\heartsuit$ (Ace), 2$\heartsuit$,. . . , 10$\heartsuit$, J$\heartsuit$ (Jack), Q$\heartsuit$ (Queen), K$\heartsuit$ (King),

A$\spadesuit$ (Ace), 2$\spadesuit$,. . . , 10$\spadesuit$, J$\spadesuit$ (Jack), Q$\spadesuit$ (Queen), K$\spadesuit$ (King),

A$\diamondsuit$ (Ace), 2$\diamondsuit$,. . . , 10$\diamondsuit$, J$\diamondsuit$ (Jack), Q$\diamondsuit$ (Queen), K$\diamondsuit$ (King)

What if you don't like card tricks?

Spend this hour productively. List all possible rearrangements of:

$$
\begin{gathered}
A\clubsuit,\ 2\clubsuit,\ \ldots,\ K\clubsuit, \\
A\heartsuit,\ 2\heartsuit, \ldots,\ K\heartsuit, \\
A\spadesuit,\ 2\spadesuit, \ldots,\ K\spadesuit, \\
A\diamondsuit,\ 2\diamondsuit,\ \ldots,\ K\diamondsuit.
\end{gathered}
$$

What if you don't like cards at all?

List all possible rearrangements of the 52 white notes on a piano, using no note twice.

By (re)arrangement–math that counts

How many ways can a collection of things be listed?

E.g., **Thelma** and **Louise**?

$$\{\textbf{Thelma, Louise}\} \text{ or } \{\textbf{Louise, Thelma}\}$$

Hence $1 + 1 = 2 \times 1 = 2$ ways.

E.g., **Martin**, **Luther** and **King**?

$$\{\textbf{M, L, K}\} \text{ or } \{\textbf{M, K, L}\} \text{ or }$$
$$\{\textbf{L, M, K}\} \text{ or } \{\textbf{L, K, M}\} \text{ or }$$
$$\{\textbf{K, M, L}\} \text{ or } \{\textbf{K, L, M}\}.$$

Hence $2 + 2 + 2 = 3 \times 2 \times 1 = 6$ ways.

By (re)arrangement–math that really counts

E.g., **John**, **Paul**, **George**, and **Ringo**?

$6+6+6+6 = 4 \times 3 \times 2 \times 1 = 24$ ways

E.g., the 10 Commandments?

$10 \times 9 \times \ldots 4 \times 3 \times 2 \times 1 = 3{,}628{,}800$ ways.

E.g., all the cards in a regular deck?

$52 \times 51 \times \ldots 4 \times 3 \times 2 \times 1$ which is a VERY LARGE number of ways.

Little Fibs

Shuffle the deck well, and have one card each selected by two spectators.

They remember their cards (value and suit), share the results with each other, and tell you the sum of the chosen card values.

You soon announce what each individual card is!

Secret Number 1:

> *When several numbers are added up, each one can be determined from the sum.*

Secret Number 2:

Totally free choices of cards are offered, but only from a controlled small part of the deck.

The possibilities are narrowed down by having six key cards at the top of the deck at the start, in any order, and keeping them there throughout some fair-looking shuffles.

The mathematics is about numbers (card values).

Secret Number 3:

You memorize the suits of the top six cards.

We use the Fibonacci numbers: start with 1, 2; add to get the next one. Repeat.

$$1 + 2 = 3,$$

$$2 + 3 = 5,$$

$$3 + 5 = 8,$$

$$5 + 8 = 13, \text{ and so on.}$$

The list of Fibonacci numbers continues forever:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .

For magic with playing cards, we focus on the first six of these, namely the **Little Fibs**, i.e., 1, 2, 3, 5, 8, and 13, agreeing that 1 = Ace and 13 = King.

Consider particular cards with those values, e.g.,

A♣, 2♡, 3♠, 5♢, 8♣, K♡ (CHaSeD order).

If any two of these are selected, then they can be determined from the sum of their values, because of the *unaddition property*:

Possible totals arise in only one way.

Little-known amazing fact:

Every positive whole number can be written in exactly one way as a sum of different *and* non-consecutive *Fibonacci numbers*

1, 2, 3, 5, 8, 13, 21, 34, ...

This was apparently not noticed before 1939, when Belgian amateur mathematician Edouard Zeckendorf did so. He published it in 1972, after he'd retired.

It's easy to break up a number into its component Fibonacci numbers: just peel off the largest possible one, then repeat as needed ("greedy algorithm".)

For instance, $6 = 5 + 1$ (not $3 + 2 + 1$ or $3 + 3$), and $10 = 8 + 2$. For the card trick, we only need to peel off the largesr possible Fib like this.

More generally, $20 = 13 + 5 + 2$, whereas, $27 = 21 + 6 = 21 + 5 + 1$. Similarly, $51 = 34 + 17 = 34 + 13 + 4 = 34 + 13 + 3 + 1$. Note, $13 = 13$.

See the "Little Fibs" video on *Numberphile*.

Twice as impressive?

Can you have four cards picked from the six Little Fibs, the total revealed, and name all four cards?

Hint: all six values add up to 32.

Suppose that four cards are chosen, and you're told the values sum to 22.

Then the two *not* chosen add up to $32 - 22 = 10$, so *they* are the 8 and 2.

Hence the chosen cards are the Ace, 3, 5, and King.

Other lists of numbers work: the Lucas sequence 2, 1, 3, 4, 7, 11, 18, ..., is a kind of generalized Fibonacci sequence. If we omit the 2, it too has the desired "unaddition" property.

We don't need generalized Fibonacci sequences.

1, 2, 4, 6, 10 works. So does 1, 2, 5, 7, 13.

And?

Can we find a set of five or six interesting numbers such that *all* subsets of any size generate unique sums?

The Three Scoop Miracle

Hand out the deck for shuffling. A spectator is asked to call out her favourite ice-cream flavour; let's suppose she says, "Chocolate."

Take the cards back, and take off about a quarter of the deck. Mix them further until told when to stop.

Deal cards, one for each letter of "chocolate," before dropping the rest on top as a topping.

This spelling/topping routine is repeated twice more—so three times total.

The Three Scoop Miracle

Emphasize how random the dealing was, since the cards were shuffled and you had no control over the named ice-cream flavour.

Have the spectator press down hard on the final top card, asking her to magically turn it into a specfic card, say the 4♠. When that card is turned over it is found to be the desired card.

Published 21 Oct 2004 at MAA.org as "Low Down Triple Dealing" as the first *Card Colm*, dedicated to recreational math populariser Martin Gardner on the occasion of his 90th birthday.

Low Down Triple Dealing

The key move here is a *reversed transfer* of a fixed number of cards in a packet—at least half—from top to bottom, done three times total.

The dealing out of k cards from a packet that runs $\{1, 2, \ldots, k-1, k, k+1, k+2, \ldots, n-1, n\}$ from the top down, and then dropping the rest on top as a unit, yields the rearranged packet

$$\{k+1, k+2, \ldots, n-1, n, k, k-1, \ldots, 2, 1\}.$$

True, but it hides what is really going on!

Low Down Triple Dealing

When $k \geq \frac{n}{2}$, doing this three times brings the original bottom card(s) to the top.

Given a flavour of length k and a number $n \leq 2k$, the packet of size n breaks symmetrically into three pieces T, M, B of sizes $n-k$, $2k-n$, $n-k$, such that the count-out-and-transfer operation (of k cards each time) is

$$T, M, B \rightarrow B, \overline{M}, \overline{T},$$

where the bar indicates reversal within that piece.

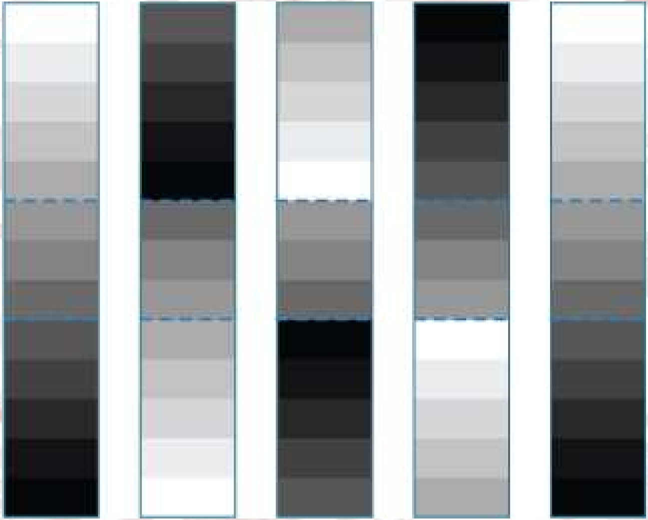
Low Down Triple Dealing Generalized?

Using this approach, the Bottom to Top (with three moves) property can be proved. Actually...

The Bottom to Top Property is only 75% of the story. Here's the real scoop:

The Period 4 Principle *If four reversed transfers of k cards are done to a packet of size n, where $k \geq \frac{n}{2}$, then every card in the packet is returned to its original position.*

Proof without words



Given Any Five Cards

The mathemagician Aodh gives a deck of cards to a spectator and asks for any five cards. Aodh glances at those, and hands one back to the spectator, who hides it. Aodh places the other four cards in a face-up row on the table.

Bea, who has witnessed and heard nothing prior to this, enters the room, glances at the cards on the table, and after a suitable pause, promptly reveals the identity of the hidden card.

This is a two person card trick. The second person —Aodh's mathematical accomplice Bea—only participates in the dramatic finale.

It's as easy as Aodh, Bea, . . . see?

This superb effect is by mathematician William Fitch Cheney Jnr (1904-1974), who received the first PhD in mathematics awarded by MIT (1927).

A keen magician all his life, Cheney was Editor of the Puzzle Section of the *American Mathematical Monthly* from 1930 to 1940.

The trick dates from the late 1940s.

Note that Aodh gets to choose which card to hand back, and then, in what exact order to place the remaining four cards.

Fitch Cheney's Five-Card Trick

Three main ideas make this magic possible:

1. At least two of the five cards are of the same suit.

For the sake of argument, assume that Aodh has two Clubs.

One Club is handed back, and by placing the remaining four cards in some particular order, Aodh effectively tells Bea the identity of the Club handed back.

2. Aodh can use one designated position (e.g., the first of the four available) for the retained Club—which determines the suit of the hidden card. This we call the suit-giver card.

Aodh used the other three positions for the placement of the remaining three cards, which can be arranged in 6 ways.

Fitch Cheney's Five-Card Trick

If Aodh and Bea agree in advance on an assignment linking the possible arrangements and the numbers $1, 2, \ldots, 6$, then Aodh can communicate one of 6 things.

Focus on the other three cards: what *can* one say about them? Not much—for instance, some or all of them could be Clubs too, or there could be other suit matches!

One thing *is* certain: they are all different from each other, so with respect to the CHaSeD ordering of the deck, one of them can be viewed as LOW, another as MEDIUM, and a third as HIGH.

$$\begin{gathered} A\clubsuit,\ 2\clubsuit,\ \ldots,\ K\clubsuit, \\ A\heartsuit,\ 2\heartsuit, \ldots,\ K\heartsuit, \\ A\spadesuit,\ 2\spadesuit, \ldots,\ K\spadesuit, \\ A\diamondsuit,\ 2\diamondsuit,\ \ldots,\ K\diamondsuit. \end{gathered}$$

Fitch Cheney's Five-Card Trick

This permits for an unambiguous and easily remembered way to communicate a number between 1 and 6.

Mentally, label the three cards L (low), M (medium), and H (high) with respect to the suggested deck ordering.

The 6 permutations of L,M,H are always ordered by rank, i.e.,

1 = LMH, 2 = LHM, 3 = MLH, 4 = MHL, 5 = HLM, 6 = HML.

Aodh places the three cards in a row accordingly, right after the suit-giver card, to communicate the desired number to Bea.

But surely 6 isn't enough? The hidden card could in general be any one of 12 other Clubs! This brings us to the third main idea:

3. Aodh must be careful as to which card he gives back.

Fitch Cheney's Five-Card Trick

Imagine the 13 card values, 1 (Ace), $2, 3, \ldots, 10$, J, Q, K, as being arranged clockwise on a circle. The suit match cards are at most 6 values apart, i.e., counting clockwise, one is "higher" and lies at most 6 vertices past the other.

Aodh gives back for hiding this "higher" valued Club, and uses the retained "lower" Club and the other three cards to communicate to Bea the identity of the hidden fifth card. Aodh is basically saying to Bea, "go k past the suit-giver, clockwise."

For example, if Aodh has the 2♣ and 8♣, he hands back the 8♣ to be hidden. However, if he has the 2♣ and J♣, he hands back the 2♣. In general, Aodh retains a card of some suit and communicates another of the same suit, which is k "higher" than the one retained, for some integer k from 1 to 6 inclusive.

If Aodh is "playing" the J♣ to communicate the 2♣, then $k = 4$, so he places the other cards in the order MHL. Bea then knows the hidden card is a Club, decodes the MHL as 4, and mentally counts 4 past the visible J♣ to get the 2♣.

Fitch Four Glory

It's customary to extend the Fitch Cheney "given any five cards" trick to larger decks—just try doing it for a regular deck with a Joker—but we now suggest a different direction to explore.

(Fitch Four) *Four random cards from a regular deck are handed to Aodh. He hides one, face down.*

He then places the remaining three cards left to right in a row on the table, some face up, some face down.

Bea, having seen nothing so far, look at the row of cards, and in due course name the hidden fourth card—even if all three cards are face down!

This seems impossible, but it can be done honestly!

Made that list of possible dealings of 52 cards, or orderings of the 52 white keys on a piano?

Checked it twice? It should have

$$52 \times 51 \times 50 \times 49 \times \ldots 3 \times 2 \times 1$$

items on it, which is

80658175170943878571660636856403766975289505440883277824000000000000.

That's approximately

$$8 \times 10^{67}$$

This number is larger than the current estimates of the total number of elementary particles in our galaxy.

That's a very big deal indeed.

Imagine two seeds released into the air from the top of a windy mountain on different days.

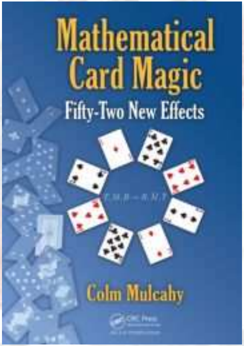
What are the chances that the seeds end up in the exact same place?

It's very very very very small.

But it's much much much much bigger than the probability that two well-shuffled decks of cards are in the same order.

Hence, a well-shuffled deck is almost certainly in an order that no other deck has ever been in before!

Cool, Colm & Collected



Mathematical Card Magic: Fifty-Two New Effects

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